Applications of the Integrals

Case Study Based Questions

Case Study 1

A bridge connects two districts 50 feet apart. The arch on the bridge is in parabolic form. The highest point on the bridge is 5 feet above the road at the middle of the bridge as shown in the figure.

Based on the above information, solve the following questions:

Q1. The equation of the parabola designed on the bridge is:

a. $y^2 = 125x$ b. $y^2 = -125x$

c. $x^2 = 125y$ d. $x^2 = -125y$

Q 2. The value of the integral $\int_{-25}^{25} \frac{x^2}{125} dx$ is:

a. $\frac{1000}{3}$ sq. units b. $\frac{250}{3}$ sq. units

c. 1200 sq. units d. 0 sq. units

Q 3. The integrand of the integral $\int_{-25}^{25} x^2 \sin x \, dx$ is .... function.

a. an even b. an odd

c. Neither odd nor even d. None of these

Q4. The area formed by the curve $y^2 = 25x$, X-axis, x = 4 and x = 9 is:

a. $\frac{100}{3}$ sq. units b. $\frac{110}{3}$ sq. units

c. $\frac{190}{3}$ sq. units d. $\frac{200}{3}$ sq. units

Q5. The area formed by the curve x^2 = 125y, Y-axis, y = 16 and y = 25 is:

a. $\frac{610\sqrt{5}}{3}$ sq. units

b. $\frac{1000}{3}$ sq. units

c. $\frac{4}{3}$ sq. units

d. None of these

Solutions

1. Since, the bridge is open downwards.

Therefore, the equation of the parabola on the bridge is $x^2 = -4ay$.

From the given options, the equation is of the form $x^2 = -125y$.

So, option (d) is correct.

2. Here integrand $f(x) = \frac{x^2}{125}$

Now, $f(-x) = \frac{(-x)^2}{125} = \frac{x^2}{125} = f(x)$

$\therefore$ $f(x)$ is an even function.

Now, $\int_{-25}^{25} \frac{x^2}{125}\,dx = 2\int_0^{25} \frac{x^2}{125}\,dx = \frac{2}{125}\left[\frac{x^3}{3}\right]_0^{25}$

$= \frac{2}{125} \times \frac{1}{3}[(25)^3 - 0]$

$= \frac{2 \times 25 \times 25 \times 25}{125 \times 3} = \frac{250}{3}$ sq. units

So, option (b) is correct.

3. Here, integrand $f(x) = x^2 \sin x$

Now, $f(-x) = (-x)^2 \sin(-x)$

$= x^2(-\sin x) = -x^2 \sin x = -f(x)$

$\therefore$ $f(x)$ is an odd function.

So, option (b) is correct.

4. Equation of curve, $y^2 = 25x$

$\therefore$ Required area $= \int_{x=4}^{x=9} y\,dx = \int_4^9 \sqrt{25x}\,dx$

$$= 5\int_4^9 x^{1/2}\, dx = 5\left[\frac{x^{3/2}}{3/2}\right]_4^9$$

$$= 5 \times \frac{2}{3}[(9)^{3/2} - (4)^{3/2}]$$

$$= \frac{10}{3}(27 - 8) = \frac{190}{3} \text{ sq. units}$$

So, option (c) is correct.

5. Equation of curve, $x^2 = 125y$

$\therefore$ Required area $= \int_{y=16}^{y=25} x\, dy$

$$= \int_{16}^{25} \sqrt{125\,y}\, dy = 5\sqrt{5}\left[\frac{y^{3/2}}{3/2}\right]_{16}^{25}$$

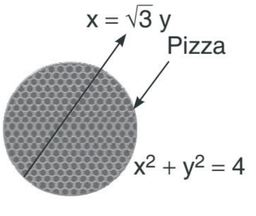
$$= 5\sqrt{5} \times \frac{2}{3}[(25)^{3/2} - (16)^{3/2}]$$

$$= \frac{10\sqrt{5}}{3}(125 - 64) = \frac{610\sqrt{5}}{3} \text{ sq. units}$$

So, option (a) is correct.

Case Study 2

A man cut a pizza with a knife, pizza is circular in shape which is represented by $x^2 + y^2 = 4$ and sharp edge of knife represents a straight line given by $x = \sqrt{3y}$

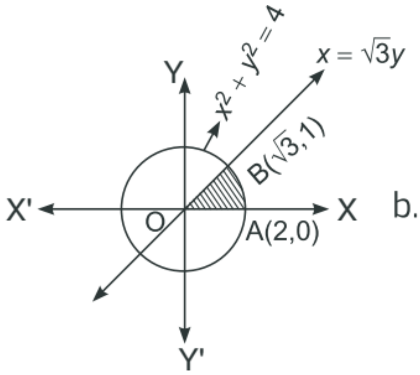


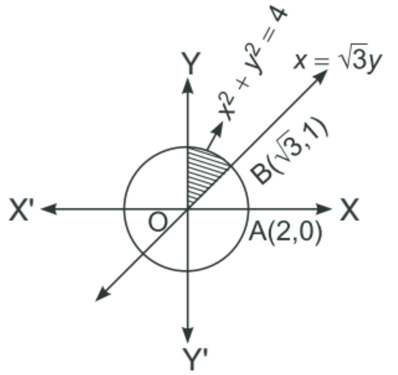
Based on the given information, solve the following questions:

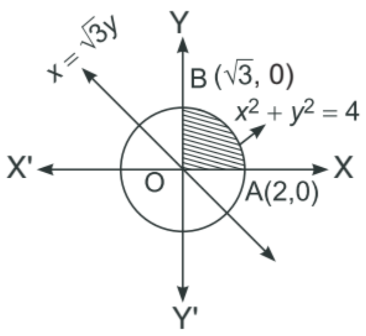
Q1. The point(s) of intersection of the edge of knife (line) and pizza shown in the figure is (are):

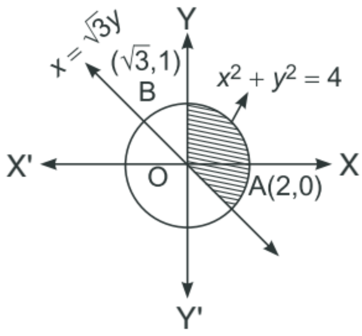
a. $(1, \sqrt{3}), (-1, -\sqrt{3})$
b. $(\sqrt{3}, 1), (-\sqrt{3}, -1)$
c. $(\sqrt{2}, 0), (0, \sqrt{3})$
d. $(-\sqrt{3}, 1), (1, -\sqrt{3})$

Q2. Which of the following shaded portion represent the smaller area bounded by pizza and edge of knife in first quadrant?

a. 

b. 

c. 

d. 

Q3. Value of area of the region bounded by circular pizza and edge of knife in first quadrant is:

a. $\frac{\pi}{2}$ sq. units   b. $\frac{\pi}{3}$ sq. units

c. $\frac{\pi}{5}$ sq. units   d. π sq. units

Q4. Area of each slice of pizza when a man cut the pizza into 4 equal pieces, is:

a. π sq. units   b. $\frac{\pi}{2}$ sq. units

c. 3π sq. units   d. 2π sq. units

Q5. Area of whole pizza is:

a. 3π sq. units   b. 2π sq. units

c. 5π sq. units   d. 4π sq. units

Solutions

1. We have, $x^2 + y^2 = 4$ ...(1)

and $x = \sqrt{3}y$ ...(2)

From eqs. (1) and (2), we get

$3y^2 + y^2 = 4 \quad \Rightarrow \quad 4y^2 = 4$

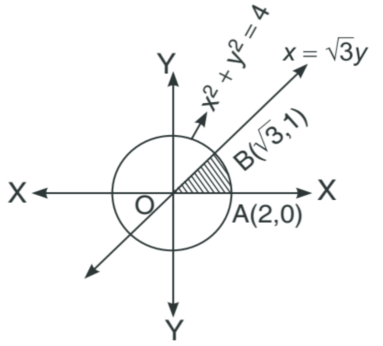
$\Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1$

From eq. (2), $x = \sqrt{3}, -\sqrt{3}$

$\therefore$ Points of intersection of pizza and edge of knife are $(\sqrt{3}, 1), (-\sqrt{3}, -1)$.

So, option (b) is correct.

2.



So, option (a) is correct.

3. Required area $= \int_0^{\sqrt{3}} \dfrac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{3}{2} - 0\right] + \left[2\sin^{-1}(1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{2\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq. units}$$

So, option (b) is correct.

4. We have, $x^2 + y^2 = 4$

$\Rightarrow$ $(x-0)^2 + (y-0)^2 = (2)^2$

$\therefore$ Radius $= 2$

Area of $\frac{1}{4}$th slice of pizza $= \frac{1}{4}\pi(2)^2 = \pi$ sq. units

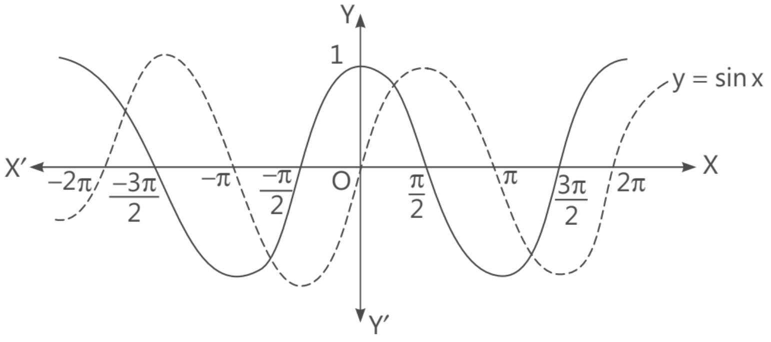
So, option (a) is correct.

5. Area of whole pizza $= \pi(2)^2 = 4\pi$ sq. units

So, option (d) is correct.

Case Study 3

In a classroom, teacher explains the properties of a particular curve by saying that this particular curve has beautiful ups and downs. It starts at 1 and heads down until π radian and then heads up again and closely related to sine function and both follow each other, exactly $\frac{\pi}{2}$ radians apart as shown in figure.



Based on the above information, solve the following questions:

Q1. Write the name of curve, about which teacher explained in the classroom.

Q2. Find the area of curve explained in the above passage from 0 to $\frac{\pi}{2}$.

Q3. Find the area of curve discussed in the above passage from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

OR

Find the area of curve discussed in the above passage from $\frac{3\pi}{2}$ to 2π.

Solutions

1. Here, teacher explained about cosine curve.
2. Required area $= \int_0^{\pi/2} \cos x \, dx$

$$= [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1 \text{ sq. unit}$$

3. Required area $= \left| \int_{\pi/2}^{3\pi/2} \cos x \, dx \right|$

$$= |[\sin x]_{\pi/2}^{3\pi/2}|$$

$$= \left| \sin\frac{3\pi}{2} - \sin\frac{\pi}{2} \right|$$

$$= |-1-1| = |-2| = 2 \text{ sq. units}$$

[Since, area can't be negative]

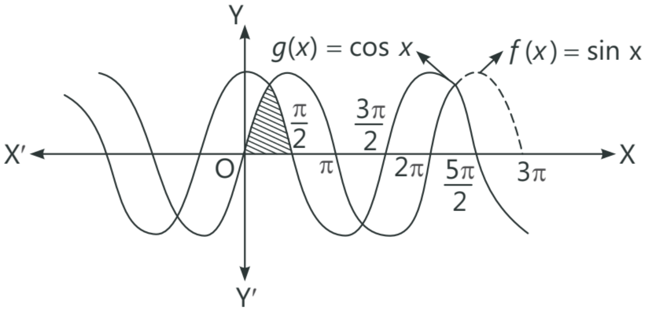
Or

Required area $= \int_{3\pi/2}^{2\pi} \cos x \, dx = [\sin x]_{3\pi/2}^{2\pi}$

$$= \sin 2\pi - \sin\frac{3\pi}{2} = 0 - (-1) = 1 \text{ sq. unit}$$

Case Study 4

Graphs of two functions $f(x) = \sin x$ and $g(x) = \cos x$ is given below:



Based on the above information, solve the following questions:

Q 1. In [0, π], the curves $f(x) = \sin x$ and $g(x) = \cos x$ intersect at $x =$

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. π

Q 2. The value of $\int_0^{\pi/4} \sin x \, dx$ is:

a. $1 - \frac{1}{\sqrt{2}}$ b. $1 + \frac{1}{\sqrt{2}}$ c. $2 - \frac{1}{\sqrt{2}}$ d. $2 + \frac{1}{\sqrt{2}}$

Q 3. The value of $\int_{\pi/4}^{\pi/2} \cos x \, dx$ is:

a. $1 + \frac{1}{\sqrt{2}}$ b. $1 - \frac{1}{\sqrt{2}}$ c. $2 - \sqrt{2}$ d. $2 + \sqrt{2}$

Q 4. The value of $\int_0^{\pi} \sin x \, dx$ is:

a. 0 b. 1
c. 2 d. –2

Q 5. The value of $\int_0^{\pi/2} \sin x \, dx$ is:

a. 0 b. 1
c. 2 d. –2

Solutions

1. For point of intersection, we have $\sin x = \cos x$

$$\Rightarrow \quad \frac{\sin x}{\cos x} = 1$$

$$\Rightarrow \quad \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

So, option (c) is correct.

2. $\int_0^{\pi/4} \sin x\, dx = [-\cos x]_0^{\pi/4}$

$$= -\cos\frac{\pi}{4} + \cos 0 = 1 - \frac{1}{\sqrt{2}}$$

So, option (a) is correct.

3. $\int_{\pi/4}^{\pi/2} \cos x\, dx = [\sin x]_{\pi/4}^{\pi/2}$

$$= \sin\frac{\pi}{2} - \sin\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}$$

So, option (b) is correct.

4. $\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi}$

$$= [-\cos\pi + \cos 0] = (1+1) = 2$$

So, option (c) is correct.

5. $\int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2}$

$$= \left[-\cos\frac{\pi}{2} + \cos 0\right] = 0 + 1 = 1$$

So, option (b) is correct.

Case Study 5

Consider the following equations of curves $x^2 = y$ and $y = x$.

Based on the above information, solve the following questions:

Q1. Find the point(s) of intersection of both the curves.

Q2. Draw the graph of area bounded by the curves.

Q 3. Find the value of the integral $\int_0^1 x\, dx$.

Or

Find the value of the integral $\int_0^1 x^2\, dx$.

Solutions

1. We have, $x^2 = y$ ...(1)

and $x = y$ ...(2)

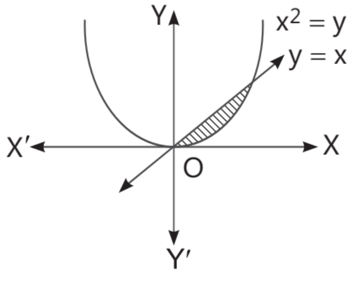
From eqs. (1) and (2),

$$x^2 = x$$
$$\Rightarrow \quad x^2 - x = 0$$
$$\Rightarrow \quad x(x-1) = 0$$
$$\Rightarrow \quad x = 0, 1$$

From eq. (2), $y = 0, 1$

Thus, required points of intersection are (0, 0), (1, 1).

2.



3. $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

Or

$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

Case Study 6

Consider the following equation of curve $y^2 = 4x$ and straight line $x + y = 3$.

Based on the above information, solve the following questions:

Q1. Write the points where the line x + y = 3 cuts the X and Y-axes.

Q2. Find the point(s) of intersection of two given curves.

Q3. Draw the graph and represent the shaded portion of area bounded by the given curves.

Or

Find the value of integral $\int_{-6}^{2} (3-y)\,dy$.

Solutions

1. Line $x + y = 3$ cuts the X-axis and Y-axis at (3, 0) and (0, 3) respectively.

[Since, at X-axis, $y = 0$ and at Y-axis, $x = 0$]

2. We have, $y^2 = 4x$ ...(1)

and $x + y = 3$ ...(2)

From eqs. (1) and (2), we have

$$y^2 = 4(3 - y)$$

$$\Rightarrow \quad y^2 + 4y - 12 = 0$$

$$\Rightarrow \quad y^2 + (6 - 2)y - 12 = 0$$

$$\Rightarrow \quad y^2 + 6y - 2y - 12 = 0$$

$$\Rightarrow \quad y(y + 6) - 2(y + 6) = 0$$

$$\Rightarrow \quad (y + 6)(y - 2) = 0$$

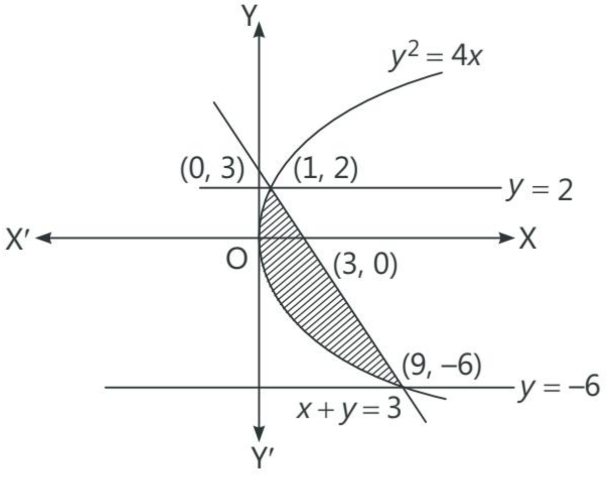
$$\Rightarrow \quad y = 2, \quad y = -6$$

From eq. (2), $x = 3 - 2 = 1$

or $x = 3 + 6 = 9$

$\therefore$ Required points of intersection are (1, 2), (9, –6).

3.



Or

$$\int_{-6}^{2} (3 - y)\, dy = \left[3y - \frac{y^2}{2}\right]_{-6}^{2}$$

$$= \left[6 - \frac{4}{2}\right] - \left[3(-6) - \frac{(-6)^2}{2}\right]$$

$$= 4 + 36 = 40$$

Case Study 7

A mirror in the shape of an ellipse is represented by $\frac{x^2}{9} + \frac{y^2}{4} = 1$ was hanging on the wall. Sanjeev and his daughter were playing with football inside the house, even his wife refused to do so. All of a sudden,

football hit the mirror and got a scratch in the shape of line represented by $\frac{x}{3}+\frac{y}{2}=1.$

Based on the above information, solve the following questions:

Q1. Find the point(s) of intersection of ellipse and scratch (straight line).

Q2. Draw the graph and show the area of smaller region bounded by the ellipse and line.

Q 3. Find the value of $\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx$.

Or

Find the value of $2\int_0^3 \left(1-\frac{x}{3}\right)dx$.

Solutions

1. We have,

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \qquad ...(1)$$

and $$\frac{x}{3}+\frac{y}{2}=1 \qquad ...(2)$$

From eq. (1), we have

$$\frac{1}{9}\cdot x^2+\frac{1}{4}\cdot\left\{2\left(1-\frac{x}{3}\right)\right\}^2=1$$

$$\Rightarrow \quad \frac{x^2}{9}+1+\frac{x^2}{9}-\frac{2x}{3}=1$$

$$\Rightarrow \quad \frac{2x^2}{9}-\frac{2x}{3}=0$$

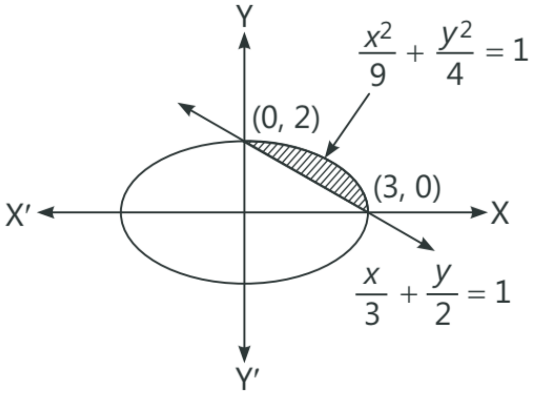
$$\Rightarrow \quad \frac{2x}{3}\left(\frac{x}{3}-1\right)=0$$

$$\Rightarrow \quad x=0, 3$$

From eq. (2); $\quad y=2, 0$

$\therefore$ Required points of intersection are (0, 2) and (3, 0).

2.



3. $\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx = \frac{2}{3}\int_0^3 \sqrt{(3)^2-x^2}\,dx$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3}{2}(0)+\frac{9}{2}\sin^{-1}(1)-\frac{1}{2}(0)-\frac{9}{2}\sin^{-1}(0)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}\right] = \frac{3\pi}{2}$$

Or

$$2\int_0^3\left(1-\frac{x}{3}\right)dx = 2\left[x-\frac{x^2}{6}\right]_0^3$$

$$= 2\left(3-\frac{9}{6}-0-0\right) = 2\times\frac{3}{2} = 3$$

Solutions for Questions 8 to 17 are Given Below

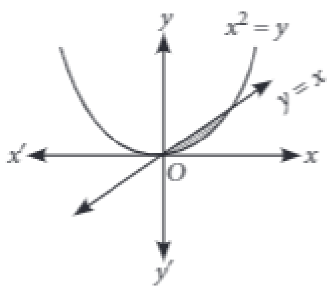
Case Study 8

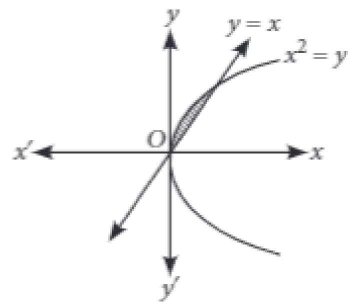
Consider the following equations of curves $x^2 = y$ and $y = x$.
On the basis of above information, answer the following questions.

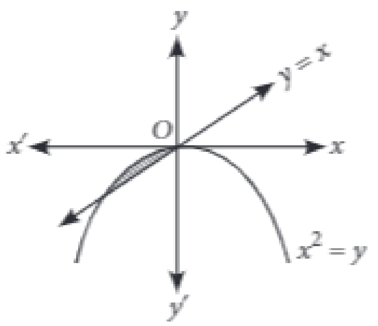
(i) The point(s) of intersection of both the curves is (are)

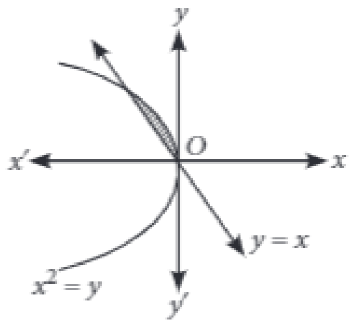
(a) (0, 0), (2, 2) (b) (0, 0), (1, 1) (c) (0, 0), (–1, –1) (d) (0, 0), (–2, –2)

(ii) Area bounded by the curves is represented by which of the following graph?

(a) 

(b) 

(c) 

(d) 

(iii) The value of the integral $\int_0^1 x\,dx$ is

(a) 1/4 (b) 1/3 (c) 1/2 (d) 1

(iv) The value of the integral $\int_0^1 x^2 dx$ is

(a) 1/4 (b) 1/3 (c) 1/2 (d) 1

(v) The value of area bounded by the curves $x^2 = y$ and $x = y$ is

(a) $\frac{1}{6}$ sq. unit (b) $\frac{1}{3}$ sq. unit (c) $\frac{1}{2}$ sq. unit (d) 1 sq. unit

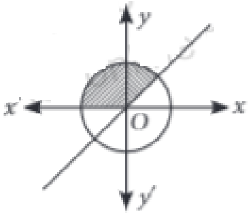
Case Study 9

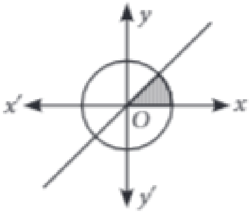
Consider the curve $x^2 + y^2 = 16$ and line $y = x$ in the first quadrant.
Based on the above information, answer the following questions.

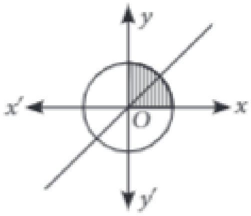
(i) Point of intersection of both the given curves is

(a) $(0, 4)$ (b) $(0, 2\sqrt{2})$ (c) $(2\sqrt{2}, 2\sqrt{2})$ (d) $(2\sqrt{2}, 4)$

(ii) Which of the following shaded portion represent the area bounded by given two curves?

(a)


(b)


(c)


(d) None of these

(iii) The value of the integral $\int_0^{2\sqrt{2}} x\,dx$ is

(a) 0 (b) 1 (c) 2 (d) 4

(iv) The value of the integral $\int_{2\sqrt{2}}^{4} \sqrt{16 - x^2}\,dx$ is

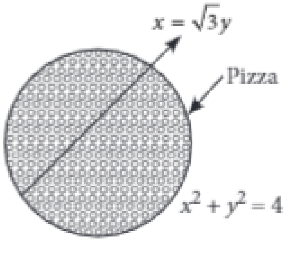
(a) $2(\pi - 2)$ (b) $2(\pi - 8)$ (c) $4(\pi - 2)$ (d) $4(\pi + 2)$

(v) Area bounded by the two given curves is

(a) 3π sq. units (b) $\frac{\pi}{2}$ sq. units (c) π sq. units (d) 2π sq. units

Case Study 10

A child cut a pizza with a knife. Pizza is circular in shape which is represented by $x^2 + y^2 = 4$ and sharp edge of knife represents a straight line given by $x = \sqrt{3}y$.
Based on the above information, answer the following questions.

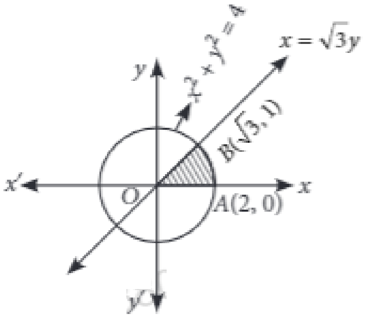


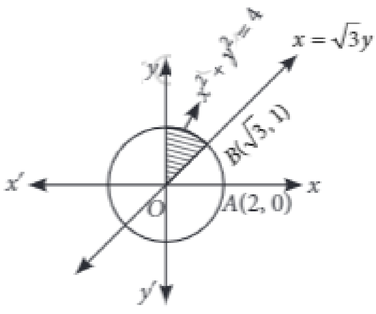
(i) The point(s) of intersection of the edge of knife (line) and pizza shown in the figure is (are)

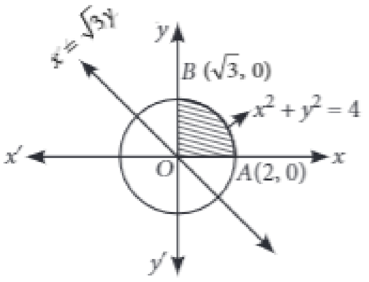
(a) $(1, \sqrt{3}), (-1, -\sqrt{3})$ (b) $(\sqrt{3}, 1), (-\sqrt{3}, -1)$

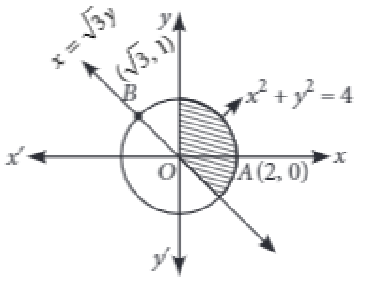
(c) $(\sqrt{2}, 0), (0, \sqrt{3})$ (d) $(-\sqrt{3}, 1), (1, -\sqrt{3})$

(ii) Which of the following shaded portion represent the smaller area bounded by pizza and edge of knife in first quadrant?

(a) 

(b) 

(c) 

(d) 

(iii) Value of area of the region bounded by circular pizza and edge of knife in first quadrant is

(a) $\frac{\pi}{2}$ sq. units (b) $\frac{\pi}{3}$ sq. units (c) $\frac{\pi}{5}$ sq. units (d) π sq. units

(iv) Area of each slice of pizza when child cut the pizza into 4 equal pieces is

(a) π sq. units (b) $\frac{\pi}{2}$ sq. units (c) 3π sq. units (d) 2π sq. units

(v) Area of whole pizza is

(a) 3π sq. units (b) 2π sq. units (c) 5π sq. units (d) 4π sq. units

Case Study 11

Consider the following equation of curve $y^2 = 4x$ and straight line $x + y = 3$.

Based on the above information, answer the following questions.

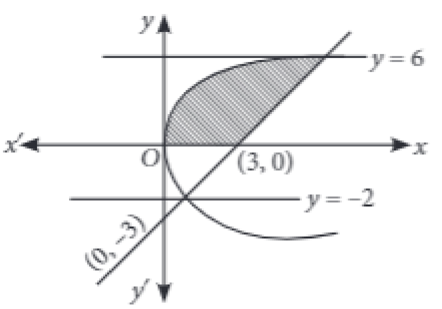
(i) The line $x + y = 3$ cuts the x-axis and y-axis respectively at

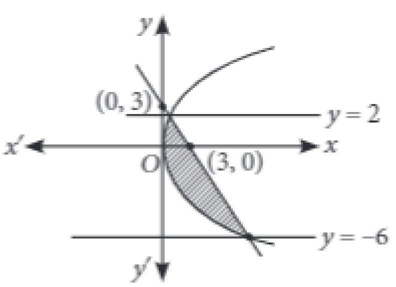
(a) (0, 2), (2, 0) (b) (3, 3), (0, 0) (c) (0, 3), (3, 0) (d) (3, 0), (0, 3)

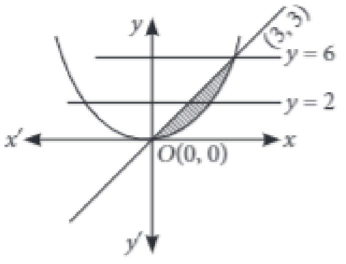
(ii) Point(s) of intersection of two given curves is (are)

(a) (1, –2), (–9, 6) (b) (2, 1), (–6, 9) (c) (1, 2), (9, –6) (d) None of these

(iii) Which of the following shaded portion represent the area bounded by given curves?

(a) 

(b) 

(c) 

(d) None of these

(iv) Value of the integral $\int_{-6}^{2} (3 - y)\,dy$ is

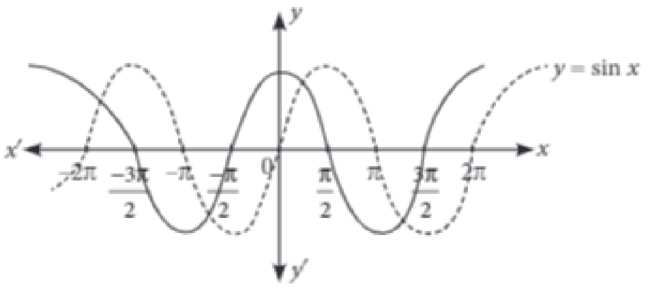
(a) 10 (b) 20 (c) 30 (d) 40

(v) Value of area bounded by given curves is

(a) 56 sq. units (b) $\frac{63}{5}$ sq. units (c) $\frac{64}{3}$ sq. units (d) 31 sq. units

Case Study 12

In a classroom, teacher explains the properties of a particular curve by saying that this particular curve has beautiful up and downs. It starts at 1 and heads down until π radian, and then heads up again and closely related to sine function and both follow each other, exactly $\frac{\pi}{2}$ radians apart as shown in figure.



Based on the above information, answer the following questions.

(i) Name the curve, about which teacher explained in the classroom.

(a) cosine (b) sine (c) tangent (d) cotangent

(ii) Area of curve explained in the passage from 0 to $\frac{\pi}{2}$ is

(a) $\frac{1}{3}$ sq. unit (b) $\frac{1}{2}$ sq. unit (c) 1 sq. unit (d) 2 sq. units

(iii) Area of curve discussed in classroom from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ is

(a) –2 sq. units (b) 2 sq. units (c) 3 sq. units (d) –3 sq. units

(iv) Area of curve discussed in classroom from $\frac{3\pi}{2}$ to 2π is

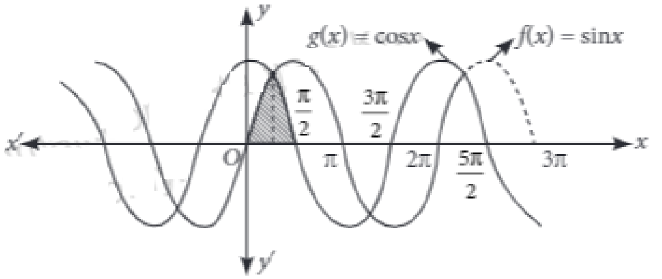
(a) 1 sq. unit (b) 2 sq. units (c) 3 sq. units (d) 4 sq. units

(v) Area of explained curve from 0 to 2π is

(a) 1 sq. unit (b) 2 sq. units (c) 3 sq. units (d) 4 sq. units

Case Study 13

Graphs of two function $f(x) = \sin x$ and $g(x) = \cos x$ is given below :



Based on the above information, answer the following questions.

(i) In $[0, \pi]$, the curves $f(x) = \sin x$ and $g(x) = \cos x$ intersect at $x =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) π

(ii) Value of $\int_0^{\pi/4} \sin x\, dx$ is

(a) $1-\frac{1}{\sqrt{2}}$ (b) $1+\frac{1}{\sqrt{2}}$ (c) $2-\frac{1}{\sqrt{2}}$ (d) $2+\frac{1}{\sqrt{2}}$

(iii) Value of $\int_{\pi/4}^{\pi/2} \cos x\, dx$ is

(a) $1+\frac{1}{\sqrt{2}}$ (b) $1-\frac{1}{\sqrt{2}}$ (c) $2-\sqrt{2}$ (d) $2+\sqrt{2}$

(iv) Value of $\int_0^{\pi} \sin x\, dx$ is

(a) 0 (b) 1 (c) 2 (d) –2

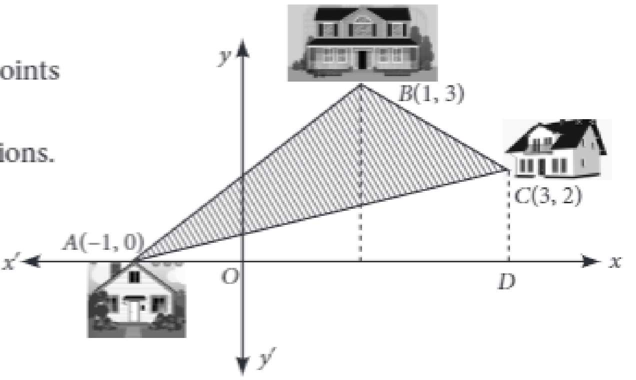
(v) Value of $\int_0^{\pi/2} \sin x\, dx$ is

(a) 0 (b) 1 (c) 3 (d) 4

Case Study 14

Location of three houses of a society is represented by the points $A(-1, 0)$, $B(1, 3)$ and $C(3, 2)$ as shown in figure.

Based on the above information, answer the following questions.



(i) Equation of line AB is

(a) $y = \frac{3}{2}(x+1)$ (b) $y = \frac{3}{2}(x-1)$

(c) $y = \frac{1}{2}(x+1)$ (d) $y = \frac{1}{2}(x-1)$

(ii) Equation of line BC is

(a) $y=\frac{1}{2}x-\frac{7}{2}$ (b) $y=\frac{3}{2}x-\frac{7}{2}$ (c) $y=\frac{-1}{2}x+\frac{7}{2}$ (d) $y=\frac{3}{2}x+\frac{7}{2}$

(iii) Area of region $ABCD$ is

(a) 2 sq. units (b) 4 sq. units (c) 6 sq. units (d) 8 sq. units

(iv) Area of ΔADC is

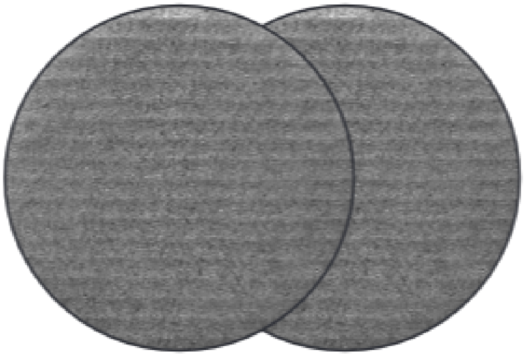
(a) 4 sq. units (b) 8 sq. units (c) 16 sq. units (d) 32 sq. units

(v) Area of ΔABC is

(a) 3 sq. units (b) 4 sq. units (c) 5 sq. units (d) 6 sq. units

Case Study 15

Ajay cut two circular pieces of cardboard and placed one upon other as shown in figure. One of the circle represents the equation $(x-1)^2+y^2=1$, while other circle represents the equation $x^2+y^2=1$.

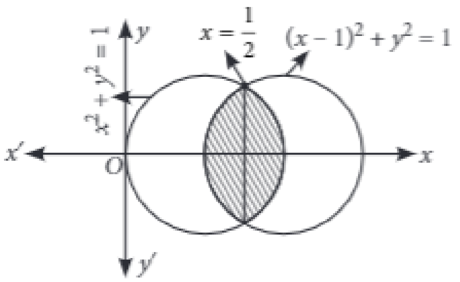


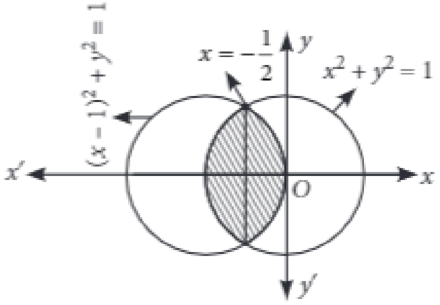
Based on the above information, answer the following questions.

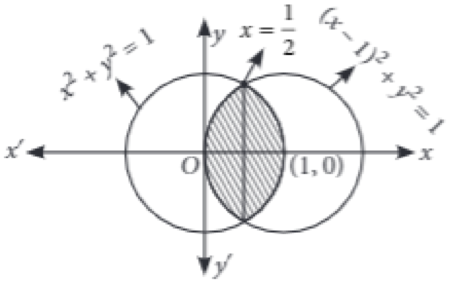
(i) Both the circular pieces of cardboard meet each other at

(a) $x=1$ (b) $x=\frac{1}{2}$ (c) $x=\frac{1}{3}$ (d) $x=\frac{1}{4}$

(ii) Graph of given two curves can be drawn as

(a)


(b)


(c)


(d) None of these

(iii) Value of $\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx$ is

(a) $\frac{\pi}{6}-\frac{\sqrt{3}}{8}$ (b) $\frac{\pi}{6}+\frac{\sqrt{3}}{8}$ (c) $\frac{\pi}{2}+\frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{2}-\frac{\sqrt{3}}{4}$

(iv) Value of $\int_{1/2}^{1} \sqrt{1-x^2}\, dx$ is

(a) $\frac{\pi}{2}+\frac{\sqrt{3}}{4}$ (b) $\frac{\pi}{6}+\frac{\sqrt{3}}{8}$ (c) $\frac{\pi}{6}-\frac{\sqrt{3}}{8}$ (d) $\frac{\pi}{2}-\frac{\sqrt{3}}{4}$

(v) Area of hidden portion of lower circle is

(a) $\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\right)$ sq. units (b) $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{8}\right)$ sq. units

(c) $\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right)$ sq. units (d) $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units

Case Study 16

A mirror in the shape of an ellipse represented by $\frac{x^2}{9}+\frac{y^2}{4}=1$ was hanging on the wall. Arun and his sister were playing with ball inside the house, even their mother refused to do so. All of sudden, ball hit the mirror and got a scratch in the shape of line represented by $\frac{x}{3}+\frac{y}{2}=1$.

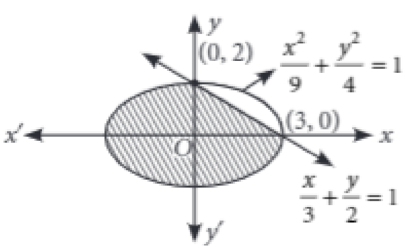
Based on the above information, answer the following questions.

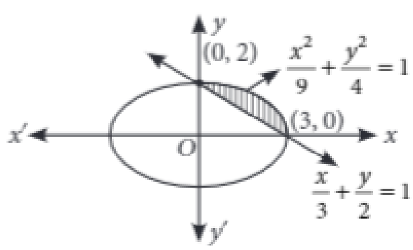
(i) Point(s) of intersection of ellipse and scratch (straight line) is (are)

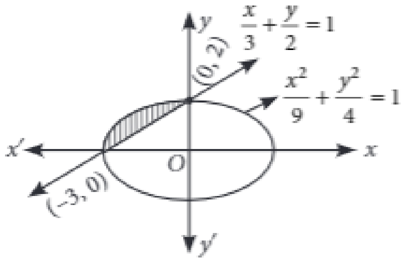
(a) (0, 2), (3, 0) (b) (2, 0), (0, 3)

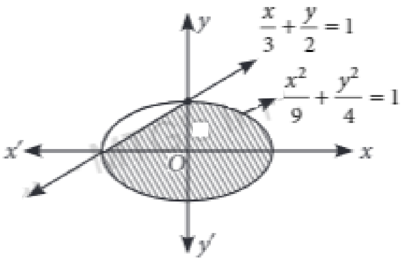
(c) (2, 3), (0, 0) (d) (0, 3), (3, 0)

(ii) Area of smaller region bounded by the ellipse and line is represented by

(a)


(b)


(c) 

(d) 

(iii) The value of $\frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx$ is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{4}$

(iv) The value of $2\int_0^3 \left(1-\frac{x}{3}\right)dx$ is

(a) 0 (b) 1 (c) 2 (d) 3

(v) Area of the smaller region bounded by the mirror and scratch is

(a) $3\left(\frac{\pi}{2}+1\right)$ sq. units (b) $\left(\frac{\pi}{2}+1\right)$ sq. units

(c) $\left(\frac{\pi}{2}-1\right)$ sq. units (d) $3\left(\frac{\pi}{2}-1\right)$ sq. units

Case Study 17

Consider the following equations of curves $y = \cos x$, $y = x + 1$ and $y = 0$.
On the basis of above information, answer the following questions.

(i) The curves $y = \cos x$ and $y = x + 1$ meet at

(a) (1, 0) (b) (0, 1) (c) (1, 1) (d) (0, 0)

(ii) $y = \cos x$ meet the x-axis at

(a) $\left(\frac{-\pi}{2}, 0\right)$ (b) $\left(\frac{\pi}{2}, 0\right)$ (c) both (a) and (b) (d) None of these

(iii) Value of the integral $\int_{-1}^{0} (x+1)\,dx$ is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{1}{3}$

(iv) Value of the integral $\int_0^{\pi/2} \cos x\,dx$ is

(a) 0 (b) –1 (c) 2 (d) 1

(v) Area bounded by the given curves is

(a) $\frac{1}{2}$ sq. unit (b) $\frac{3}{2}$ sq. units (c) $\frac{3}{4}$ sq. unit (d) $\frac{1}{4}$ sq. unit

8. **(i) (b):** We have, $x^2 = y$ ...(i) and $x = y$ ...(ii)

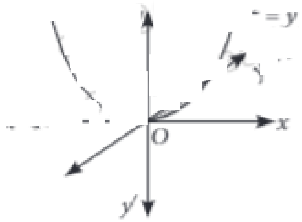
From (i) and (ii), $x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$

From (ii), $y = 0, 1$

Required points of intersection are (0, 0), (1, 1).

(ii) (a):



(iii) (c): $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

(iv) (b): $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

(v) (a): Required area $= \int_0^1 x\,dx - \int_0^1 x^2\,dx$

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units.

9. **(i) (c):** We have, $x^2 + y^2 = 16$ ...(i)

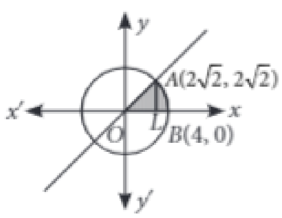
and $y = x$ ...(ii)

From (i) and (ii), $2x^2 = 16 \Rightarrow x^2 = 8 \Rightarrow x = 2\sqrt{2}$

($\because x$ lies in first quadrant)

$\therefore$ Point of intersection of (i) and (ii) in first quadrant is $(2\sqrt{2}, 2\sqrt{2})$.

(ii) (b): The shaded region which represent the area bounded by two given curves in first quadrant is shown below.



(iii) (d): $\int_0^{2\sqrt{2}} x\,dx = \left[\frac{x^2}{2}\right]_0^{2\sqrt{2}} = \frac{(2\sqrt{2})^2}{2} = \frac{8}{2} = 4$

(iv) (a): $\int_{2\sqrt{2}}^{4} \sqrt{16-x^2}\,dx = \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{2\sqrt{2}}^{4}$

$= 8\sin^{-1}(1) - 4 - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= 8\left(\frac{\pi}{2}\right) - 4 - 8\left(\frac{\pi}{4}\right) = 4\pi - 4 - 2\pi = 2\pi - 4 = 2(\pi - 2)$

(v) (d): Required area = Area (OLA) + Area (BAL)

$= \int_0^{2\sqrt{2}} x\,dx + \int_{2\sqrt{2}}^{4} \sqrt{16-x^2}\,dx$

$= 4 + 2(\pi - 2) = 2\pi$ sq. units.

10. **(i) (b):** We have, $x^2 + y^2 = 4$ ...(i)

and $x = \sqrt{3}y$ ...(ii)

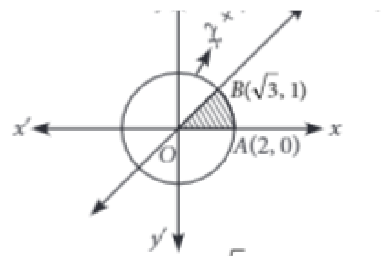
From (i) and (ii), we get

$3y^2 + y^2 = 4 \Rightarrow 4y^2 = 4 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

From (ii), $x = \sqrt{3}, -\sqrt{3}$

$\therefore$ Points of intersection of pizza and edge of knife are $(\sqrt{3}, 1), (-\sqrt{3}, -1)$.

(ii) (a):



(iii) (b): Required area $= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$

$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$

$= \frac{1}{\sqrt{3}}\left[\frac{3}{2} - 0\right] + \left[2\sin^{-1}(1) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{\sqrt{3}}{2} + \frac{2\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3}$ sq. units

(iv) (a): We have, $x^2 + y^2 = 4$

$\Rightarrow (x-0)^2 + (y-0)^2 = (2)^2$

$\therefore$ Radius = 2

Area of $\frac{1}{4}$th slice of pizza $= \frac{1}{4}\pi(2)^2 = \pi$ sq. units

(v) (d): Area of whole pizza $= \pi(2)^2 = 4\pi$ sq. units

11. **(i) (d):** Line $x + y = 3$ cuts the x-axis and y-axis at (3, 0) and (0, 3) respectively.

[Since, at x-axis, $y = 0$ and at y-axis, $x = 0$]

(ii) **(c)** : We have, $y^2 = 4x$ ...(i)

and $x + y = 3$ ...(ii)

From (i) and (ii), we have $y^2 = 4(3 - y)$

$\Rightarrow y^2 + 4y - 12 = 0 \Rightarrow y^2 + 6y - 2y - 12 = 0$

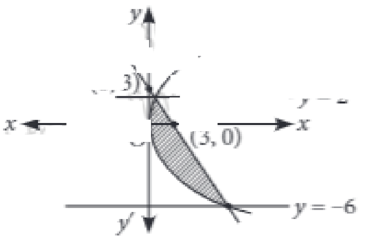
$\Rightarrow y(y + 6) - 2(y + 6) = 0$

$\Rightarrow (y + 6)(y - 2) = 0 \Rightarrow y = 2, y = -6$

From (ii), $x = 3 - 2 = 1$ or $x = 3 + 6 = 9$

$\therefore$ Required points of intersection are (1, 2), (9, -6)

(iii) **(b)**:



(iv) **(d)**: $\int_{-6}^{2}(3 - y)\,dy = \left[3y - \frac{y^2}{2}\right]_{-6}^{2}$

$= \left[6 - \frac{4}{2} - \left[3(-6) - \frac{(-6)^2}{2}\right]\right] = 4 + 36 = 40$

(v) **(c)** : Required area $= \int_{-6}^{2}(3 - y)\,dy - \int_{-6}^{2}\frac{y^2}{4}\,dy$

$= 40 - \frac{1}{4}\left[\frac{y^3}{3}\right]_{-6}^{2} = 40 - \frac{1}{4}\left[\frac{8}{3} - \frac{(-6)^3}{3}\right]$

$= 40 - \frac{2}{3} - \frac{216}{12} = \frac{480 - 8 - 216}{12} = \frac{256}{12} = \frac{64}{3}$ sq. units

12. **(i)** **(a)** : Here, teacher explained about cosine curve.

(ii) **(c)** : Required area $= \int_{0}^{\pi/2}\cos x\,dx$

$= [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$ sq. unit

(iii) **(b)**: Required area $= \left|\int_{\pi/2}^{3\pi/2}\cos x\,dx\right| = \left|[\sin x]_{\pi/2}^{3\pi/2}\right|$

$= \left|\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right| = |-1 - 1| = |-2|$

= 2 sq. units [Since, area can't be negative]

(iv) **(a)** : Required area $= \int_{3\pi/2}^{2\pi}\cos x\,dx = [\sin x]_{3\pi/2}^{2\pi}$

$= \sin 2\pi - \sin\frac{3\pi}{2} = 0 - (-1) = 1$ sq. unit

(v) **(d)**: Required area

$= \int_{0}^{\pi/2}\cos x\,dx + \left|\int_{\pi/2}^{3\pi/2}\cos x\,dx\right| + \int_{3\pi/2}^{2\pi}\cos x\,dx$

= 1 + 2 + 1 = 4 sq. units

13. **(i)** **(c)** : For point of intersection, we have

$\sin x = \cos x$

$\Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

(ii) **(a)**: $\int_{0}^{\pi/4}\sin x\,dx = [-\cos x]_0^{\pi/4} = -\cos\frac{\pi}{4} + \cos 0$

$= 1 - \frac{1}{\sqrt{2}}$

(iii) **(b)**: $\int_{\pi/4}^{\pi/2}\cos x\,dx = [\sin x]_{\pi/4}^{\pi/2} = \sin\frac{\pi}{2} - \sin\frac{\pi}{4}$

$= 1 - \frac{1}{\sqrt{2}}$

(iv) **(c)** : $\int_{0}^{\pi}\sin x\,dx = [-\cos x]_0^{\pi} = [-\cos\pi + \cos 0] = 2$

(v) **(b)**: $\int_{0}^{\pi/2}\sin x\,dx = [-\cos x]_0^{\pi/2} = \left[-\cos\frac{\pi}{2} + \cos 0\right]$

$= 0 + 1 = 1$

14. **(i)** **(a)** : Equation of line AB is

$y - 0 = \frac{3 - 0}{1 + 1}(x + 1) \Rightarrow y = \frac{3}{2}(x + 1)$

(ii) **(c)** : Equation of line BC is $y - 3 = \frac{2 - 3}{3 - 1}(x - 1)$

$\Rightarrow y = -\frac{1}{2}x + \frac{1}{2} + 3 \Rightarrow y = \frac{-1}{2}x + \frac{7}{2}$

(iii) **(d)**: Area of region $ABCD$

= Area of ΔABE + Area of region $BCDE$

$= \int_{-1}^{1}\frac{3}{2}(x + 1)\,dx + \int_{1}^{3}\left(\frac{-1}{2}x + \frac{7}{2}\right)dx$

$= \frac{3}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1} + \left[\frac{-x^2}{4} + \frac{7}{2}x\right]_{1}^{3}$

$= \frac{3}{2}\left[\frac{1}{2} + 1 - \frac{1}{2} + 1\right] + \left[\frac{-9}{4} + \frac{21}{2} + \frac{1}{4} - \frac{7}{2}\right]$

= 3 + 5 = 8 sq. units

(iv) **(a)** : Equation of line AC is $y - 0 = \frac{2 - 0}{3 + 1}(x + 1)$

$\Rightarrow y = \frac{1}{2}(x + 1)$

$\therefore$ Area of $\Delta ADC = \int_{-1}^{3}\frac{1}{2}(x + 1)\,dx = \left[\frac{x^2}{4} + \frac{1}{2}x\right]_{-1}^{3}$

$= \frac{9}{4} + \frac{3}{2} - \frac{1}{4} + \frac{1}{2} = 4$ sq. units

(v) **(b)**: Area of ΔABC = Area of region $ABCD$ – Area of ΔACD = 8 – 4 = 4 sq. units

15. (i) (b): We have, $(x-1)^2+y^2=1$

$\Rightarrow \quad y=\sqrt{1-(x-1)^2}$ ...(i)

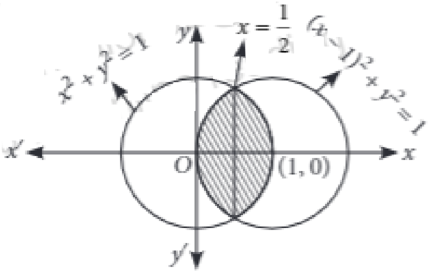
Also, $x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$ ...(ii)

From (i) and (ii), we get

$$\sqrt{1-(x-1)^2}=\sqrt{1-x^2}$$

$\Rightarrow (x-1)^2=x^2 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$

(ii) (c):



(iii) (a): $\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx$

$=\left[\frac{x-1}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2}$

$=\frac{1}{2}\left(\frac{1}{2}-1\right)\sqrt{1-\frac{1}{4}}+\frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)(0)-\frac{1}{2}\sin^{-1}(-1)$

$=\left[\frac{-1}{4}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\pi}{6}+0+\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{-\sqrt{3}}{8}-\frac{\pi}{12}+\frac{\pi}{4}$

$=\frac{\pi}{6}-\frac{\sqrt{3}}{8}$

(iv) (c): $\int_{1/2}^{1}\sqrt{1-x^2}\,dx=\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_{1/2}^{1}$

$=0+\frac{1}{2}\sin^{-1}(1)-\frac{1}{4}\sqrt{1-\frac{1}{4}}-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{4}-\frac{\sqrt{3}}{8}-\frac{\pi}{12}=\frac{\pi}{6}-\frac{\sqrt{3}}{8}$

(v) (d): Required area

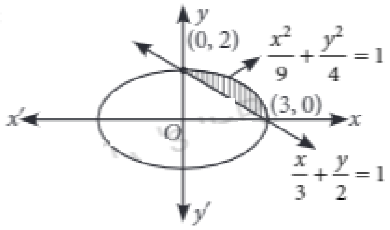
$=2\left[\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx+\int_{1/2}^{1}\sqrt{1-x^2}\,dx\right]$

$=2\left[\frac{\pi}{6}-\frac{\sqrt{3}}{8}+\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right]$

$=2\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right]=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units

16. (i) (a): Points (0, 2) and (3, 0) pass through both line and ellipse.

(ii) (b):



(iii) (c): $\frac{2}{3}\int_0^3\sqrt{9-x^2}\,dx=\frac{2}{3}\int_0^3\sqrt{(3)^2-x^2}\,dx$

$=\frac{2}{3}\left[\frac{1}{2}x\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$=\frac{2}{3}\left[\frac{3}{2}\sqrt{0}+\frac{9}{2}\sin^{-1}(1)-\frac{1}{2}(0)-\frac{9}{2}\sin^{-1}(0)\right]$

$=\frac{2}{3}\left[\frac{9}{2}\cdot\frac{\pi}{2}\right]=\frac{3\pi}{2}$

(iv) (d): $2\int_0^3\left(1-\frac{x}{3}\right)dx=2\left[x-\frac{x^2}{6}\right]_0^3$

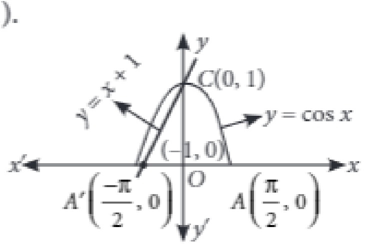
$=2\left(3-\frac{9}{6}-0-0\right)=2\times\frac{3}{2}=3$

(v) (d): Area of smaller region bounded by the mirror and scratch

$=\frac{2}{3}\cdot\int_0^3\sqrt{9-x^2}\,dx-2\int_0^3\left(1-\frac{x}{3}\right)dx$

$=\frac{3\pi}{2}-3=3\left(\frac{\pi}{2}-1\right)$ sq. units

17. (i) (b): Curves $y=\cos x$ and $y=x+1$ meet at point $C(0, 1)$.



(ii) (c): Curve $y=\cos x$ meet the x-axis at $A'\left(\frac{-\pi}{2}, 0\right)$ and $A\left(\frac{\pi}{2}, 0\right)$.

(iii) (a): $\int_{-1}^{0}(x+1)\,dx=\left[\frac{x^2}{2}+x\right]_{-1}^{0}=0-\left(\frac{1}{2}-1\right)=\frac{1}{2}$

(iv) (d): $\int_0^{\pi/2}\cos x\,dx=[\sin x]_0^{\pi/2}=\sin\frac{\pi}{2}-\sin 0=1$

(v) (b): Required area $=\int_{-1}^{0}(x+1)\,dx+\int_0^{\pi/2}\cos x\,dx$

$=\frac{1}{2}+1=\frac{3}{2}$ sq. units